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HOMOGENEOUS SINGULAR VORTEX

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An analytical description is given to the spherical partially invariant solution of the gas-dynamics equations in the case of additional symmetry — the homogeneous singular vortex. The solution was specified by a generalized potential — an auxiliary function satisfying the inhomogeneous Schwarz equation. It is proved that the part of the factor system of the homogeneous singular vortex in a Lagrangian representation that describes the kinematics of a gas particle is a system of linear equations with the potential defined by the solution of the Schwarz equation. For particular values of the adiabatic exponent equal to 1, 4/3, *and* 5/3, *the solution of the Schwarz equation is written in terms of lower-order equations. The isothermal gas flow in the homogeneous singular vortex is described. It is shown that a periodic geometrical trajectory configuration can exist but the gas density in this case has a singularity. A physically definite solution exists on time intervals that do not contain singularity points. Examples of motion obtained by implementation of analytical formulas on a computer are given.*

Key words: *spherically partially invariant solutions, homogeneous singular vortex, Schwarz equation, periodic configurations.*

Introduction. Exact solutions in fluid dynamics generated by a rotation group are of great interest by virtue of their high symmetry. The classical spherically symmetric solutions and their applications to the solution of concrete gas-dynamic problems are described in many papers (see, e.g., [1–3]).

Ovsyannikov [4] found a new class of solutions generated by the rotation group $SO(3)$ and called it the singular vortex (SV). These solutions belong to the regular partially invariant solutions (RPIS) of the equations of gas dynamics (or hydrodynamics). Part of the functions, namely, the radial velocity component and the modulus of the tangential velocity component and all thermodynamic parameters (pressure, density, and entropy) are spherically symmetric; i.e., they are invariants of the rotation group in the space of independent variables and velocities $\mathbb{R}^6(\boldsymbol{x}, \boldsymbol{u})$. However, in contrast to the spherically symmetric solutions, the velocity has a nonzero tangential component and the angle of its deflection from the meridian – the quantity $\omega = \omega(t, r, \theta, \varphi)$ is a function of all independent variables. In the language of the group analysis of differential equations, ω is called a superfluous function [5]. Thus, the SV is a RPIS of defect 1 and rank 2; the invariant independent variables are time t and the modulus of the radius vector $r = \sqrt{x^2 + y^2 + z^2}$. Ovsyannikov [4] proved the existence of solutions of this form, obtained some properties, and gave a number of examples. The description of the radial gas flow, i.e., the analysis of the invariant system of the complete factor system was considered as one of the major problems of the further study of solutions of this type. Investigation of the SV was continued in [6], where the special property of the Jacobian matrix of the SV velocity field was proved and two invariant SV submodels, i.e., solutions possessing additional symmetry besides the rotation group, were investigated. The SV is a meaningful class of physically interesting solutions, and its analysis is rather complicated.

The present study is a continuation of analytical studies of the homogeneous singular vortex (HSV) — the SV that possesses additional symmetry with respect to a certain dilatation group.

A determining property of the RPIS is that they can be described in terms of the solution of the key equation for an auxiliary function — a peculiar analog of the solution potential [7]. All parameters of the solution

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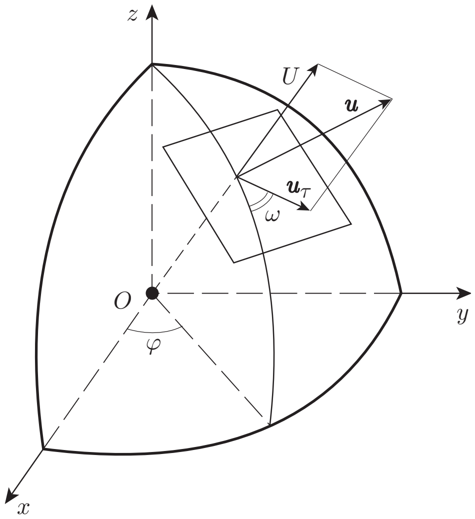

Fig. 1. Representation of the velocity vector in the singular vortex.

are determined via this function and its derivatives. For the HSV, the key equation is the inhomogeneous Schwarz equation (SE).

1. Singular Vortex Model [4]. To describe the SV model, we introduce spherical coordinates (r, θ, φ) and use polar coordinates (H, ω) for the tangential velocity component $\boldsymbol{u}_\tau = (V, W)$:

$$V = H\cos\omega, \qquad W = H\sin\omega. \tag{1.1}$$

In (1.1), $H = |\boldsymbol{u}_\tau| = \sqrt{V^2 + W^2}$ and ω is the angle formed by the vector $\boldsymbol{u}_\tau$ with the meridian (Fig. 1).

The SV is a regular partially invariant (2,1) type solution of the equations of gas dynamics (EGD) with the following representation of the solution:

$$\begin{gathered} U = U(t, r), \qquad H = H(t, r), \\ \rho = \rho(t, r), \qquad p = p(t, r), \qquad \omega = \omega(t, r, \theta, \varphi). \end{gathered} \tag{1.2}$$

Here the invariant independent variables are r and t, the invariant functions are the thermodynamic variables ρ, p, and S, the radial velocity component is denoted by U, and the modulus of the tangential component H. Th superfluous function depending on all independent variables is the angle ω. In studying SV, it is assumed that $H \not\equiv 0$; otherwise, the SV becomes the classical spherically symmetric solution.

According to the general scheme [5], for RPIS (1.2) the equations of gas dynamics split into two subsystems:
— the invariant subsystem

$$\begin{gathered} D_0 U + \rho^{-1} p_r = r^{-1} H^2, \\ D_0(rH) = 0, \qquad D_0 S = 0, \qquad p = f(\rho, S), \end{gathered} \tag{1.3}$$

where $D_0 = \partial_t + U\,\partial_r$ and the function f specifies the gas law);
— the overdetermined system for the superfluous function

$$\begin{gathered} k\sin\theta D_0\omega + \sin\theta\cos\omega\omega_\theta + \sin\omega\omega_\varphi = -\cos\theta\sin\omega, \\ \sin\theta\sin\omega\omega_\theta - \cos\omega\omega_\varphi = \cos\theta\cos\omega + h\sin\theta, \end{gathered} \tag{1.4}$$

in which the invariant functions

$$k = r/H, \qquad h = k(D_0 \ln\rho + r^{-2}(r^2 U)_r) \tag{1.5}$$

are introduced. The consistency relations for system (1.4) are described in terms of (1.5)

$$kD_0 h = h^2 + 1 \tag{1.6}$$

and complete the invariant subsystem (1.3) to a closed system.

System (1.4) was integrated in closed form for solutions of system (1.3), (1.6) in [4]. Its general solution is specified by the relation

$$F(\xi, \eta, \zeta) = 0, \tag{1.7}$$

where F is an arbitrary smooth function and ξ is an invariant Lagrangian variable

$$D_0\xi = 0, \tag{1.8}$$

the quantity η is given by

$$\eta = \cos\tau \sin\theta \cos\omega - \sin\tau \cos\theta, \tag{1.9}$$

and ζ is specified implicitly by the relation

$$\sqrt{1-\eta^2}\,\sin(\zeta + \varphi) = \cos\tau\cos\theta\cos\omega + \sin\tau\sin\theta. \tag{1.10}$$

The function τ is defined by the formula $h = \tan\,\tau$, so that $kD_0\tau = 1$. The primal problem is to study system (1.3), (1.5), (1.6), which describes the radial gas flow.

The Jacobian matrix of the SV velocity field

$$J = (\nabla_i u^j) \tag{1.11}$$

calculated in spherical coordinates so that ∇_i are covariant derivatives with respect to r, θ, and φ has algebraic invariants and eigenvalues that depend only on the invariant variables t and r. From this it follows that the SV is generated by special initial data for which the Jacobian matrix (1.11) calculated for $t = 0$ has algebraic invariants and eigenvalues that depend only on r, while the initial velocity components $V_0 = V|_{t=0}$ and $W_0 = W|_{t=0}$ depend, generally speaking, on all independent variables r, θ, and φ.

2. Homogeneous Singular Vortex [6]. The SV model admits a certain symmetry group. It is possible to construct its invariant solutions under this group. We consider the invariant submodel generated by the algebra $L_4 = \langle so(3), K\rangle$, where the dilatation operator K has the form

$$K = r\,\partial_r + U\,\partial_U + H\,\partial_H + \alpha\rho\,\partial_\rho + (\alpha+2)p\,\partial_p \tag{2.1}$$

with an arbitrary real parameter α. This algebra L_4 specifies a (1,1) type RPIS, where the invariant independent variable is time t. The corresponding submodel can be constructed by two methods: in one step, as the (1,1) type RPIS of the equations of gas dynamics, or in two steps, considering the K-invariant submodel of the SV. The result will be the same; in this case, the Lie–Ovsyannikov–Talyshev (LOT) lemma [8] on the equivalence of the multistep and single-step algorithms for constructing the solutions is extended to the partially invariant solutions [6].

The representation L_4 — the RPIS defined by (2.1) — has the form

$$U = A(t)r, \qquad H = C(t)r, \qquad \rho = r^\alpha R(t),$$
$$p = r^{\alpha+2}P(t), \qquad c^2 = \gamma r^2 B(t), \tag{2.2}$$

where $c^2 = \gamma p/\rho$ is the squared velocity of sound. In (2.2), the functions A, B, C, P, and R are expressed in terms of the auxiliary function $h = h(t)$ (a peculiar potential of the solution) and its derivatives by the formulas

$$A = -\frac{1}{2}\Big(\ln\frac{|h'|}{1+h^2}\Big)', \qquad B = B_0(1+h^2)^{-\gamma}|h'|^{(3\gamma-1)/2},$$
$$C = (1+h^2)^{-1}h', \qquad R = R_0(1+h^2)^{-(\alpha+2)/2}|h'|^{(\alpha+3)/2}, \tag{2.3}$$

where $\gamma \geqslant 1$ is the adiabatic exponent, $B_0, R_0 > 0$ are constants, and $B = R^{-1}P$.

The function h satisfies the inhomogeneous Schwarz equation

$$\{h\} \equiv \frac{h'''}{h'} - \frac{3}{2}\Big(\frac{h''}{h'}\Big)^2 = \beta_0\,\frac{|h'|^{(3\gamma-1)/2}}{(1+h^2)^\gamma}, \tag{2.4}$$

where

$$\beta_0 = 2(\alpha+2)B_0. \tag{2.5}$$

Below, along with h, we use the function τ linked to the former by the relation

$$h = \tan\,\tau, \qquad \tau = \arctan\,h. \tag{2.6}$$

In terms of τ, the representation (2.3) takes the form

$$A = -\tau''/(2\tau'), \qquad B = B_0|\cos\tau|^{1-\gamma}|\tau'|^{(3\gamma-1)/2},$$
$$C = \tau', \qquad R = R_0|\cos\tau|^{-1}|\tau'|^{(\alpha+3)/2}. \tag{2.7}$$

The replacement (2.6) transforms the Schwarz equation (2.4) into a Schwarz equation with a converted right side:

$$\{\tau\} = \beta_0|\tau'|^{(3\gamma-1)/2}/|\cos\tau|^{\gamma-1} - 2\tau'^2. \tag{2.8}$$

The algebraic invariants k_i and the eigenvalues λ_i $(i = 1, 2, 3)$ of the Jacobian matrix (1.11) of the HSV velocity field depend only on time and are defined by the formulas

$$k_1 = 3A - hC, \qquad k_2 = 3A^2 - 2hAC + C^2, \qquad k_3 = A^3 - hA^2C + C^2A - hC^3,$$
$$\lambda_1 = A - hC, \qquad \lambda_{2,3} = A \pm iC. \tag{2.9}$$

It is interesting that according to (2.9), the matrix J has complex conjugate eigenvalues.

Thus, the investigation of the HSV reduces to studying the solution of the inhomogeneous SE (2.4). Below (see Sec. 7) we give a number of cases where it can be effectively integrated but we first describe the kinematics of HSV.

3. The HSV equations in Lagrangian coordinates. From formulas (2.2) for the HSV, it follows that

$$\rho^{-1}\nabla p = (\alpha + 2)B\boldsymbol{x}. \tag{3.1}$$

Substituting the representation (3.1) into the momentum equations of the EGD in a Lagrangian description, we obtain the system of linear ordinary equations

$$\frac{d^2\boldsymbol{x}}{dt^2} + q(t)\boldsymbol{x} = 0 \tag{3.2}$$

with the potential

$$q(t) = (\alpha + 2)B(t). \tag{3.3}$$

System (3.2) is supplemented by the initial data

$$\boldsymbol{x}\Big|_{t=0} = \boldsymbol{x}_0, \qquad \frac{d\boldsymbol{x}}{dt}\Big|_{t=0} = \boldsymbol{u}_0(\boldsymbol{x}), \tag{3.4}$$

which define the initial gas particle distribution and the velocity field ($\boldsymbol{x}_0$ are the Lagrangian coordinates of the gas particles). From the special property of the Jacobian matrix J_0 for the HSV described in Secs. 1 and 2, it follows that its algebraic invariants and eigenvalues (2.9) at $t = 0$ are constants.

System (3.2) is supplemented by the continuity and energy equation, which hold by virtue of the representation of solution (2.2) and Eq. (2.4). Indeed, by virtue of (2.9), $\operatorname{div}\boldsymbol{u} = k_1(t)$ in these equations and they are integrated resulting in the representation (2.2) [6].

Lemma 1. *The kinematics of the HSV is described by a system of linear equations* (3.2) *with initial data* (3.4)*. The potential* (3.3) *and the thermodynamic parameters of the solution are defined by the representation* (2.2) *and Eq.* (2.4).

We note that there is an analogy to the HSV in an ideal fluid [9].

4. Integral of Eqs. (3.2).

Lemma 2. *System* (3.2) *has the integral*

$$\boldsymbol{x} \times \frac{d\boldsymbol{x}}{dt} = \boldsymbol{M}_0, \qquad \boldsymbol{M}_0 = \boldsymbol{x}_0 \times \boldsymbol{u}_0. \tag{4.1}$$

The motion of each gas particle in the HSV *occurs in the plane* Π *specified by the vector* $\boldsymbol{M}_0$.

Proof. We calculate the derivative of the vector $\boldsymbol{M}_0$ by virtue of Eqs. (4.1)

$$\frac{d\boldsymbol{M}_0}{dt} = \frac{d}{dt}\Big(\boldsymbol{x} \times \frac{d\boldsymbol{x}}{dt}\Big) = \frac{d\boldsymbol{x}}{dt} \times \frac{d\boldsymbol{x}}{dt} + \boldsymbol{x} \times \frac{d^2\boldsymbol{x}}{dt^2} = -q(t)\boldsymbol{x} \times \boldsymbol{x} = 0. \tag{4.2}$$

There is an obvious analogy to the area integral in classical particle mechanics [10]. We point out that each particle moves in its own plane and there is separation of the space of events into such planes.

5. Reducing the Equations of Motion to the Ermakov Equation. The indicated analogy to classical mechanics noted above can be further extended. However, in contrast to classical mechanics, gas dynamics provides a description of an individual gas particle. Next, one needs a description of the coordinated motion of a continuous medium consisting of an ensemble of such particles. In addition, the description of the solution does not reduce only to system (2.2) — the thermodynamics of the medium is specified by the representation (2.3).

By rotation in the space $\mathbb{R}^3(\boldsymbol{x})$, we reduce the plane Π of motion of the selected gas particle to the form $z = 0$. Then, the integral (4.1) becomes

$$\Pi: \quad z = 0, \quad xy' - yx' = l_0, \tag{5.1}$$

where (x, y) are Cartesian coordinates in the plane Π and $l_0 = l_0(\boldsymbol{x})$ is a function of Lagrangian coordinates.

We note that (4.1) has the form of (5.1) only for one selected particle of the gas.

In the plane Π, we convert to the polar coordinates

$$x = r\cos\psi, \qquad y = r\sin\psi. \tag{5.2}$$

We note that r coincides with the quantity $r = \sqrt{x^2 + y^2 + z^2}$ (invariance under rotation). Then, Eqs. (3.2) reduce to the following equations:

$$r'' - r\psi'^2 + qr = 0, \qquad r\psi'' + 2r'\psi' = 0. \tag{5.3}$$

The second equation (5.3) is integrated resulting in

$$r^2\psi' = l_0. \tag{5.4}$$

This is an area integral from classical mechanics [10]. The substitution ψ' from (5.4) into the first equation (5.3) results in the Ermakov equation [11]

$$r'' + q(t)r = l_0^2/r^3 \tag{5.5}$$

with the potential (3.3).

Lemma 3. *The kinematics of the HSV in the plane of motion of the gas particle specified by the equation $z = 0$ is described by Eqs.* (5.4) *and* (5.5).

This result is a direct consequence of the mechanical analogy. The following statement is less obvious.

Lemma 4. *The system of equations* (5.4) *and* (5.5) *is equivalent to the Schwarz equation* (2.8).

Proof. We express r from (5.4). We have (assuming that $l_0 > 0$ and $\psi' > 0$)

$$\begin{gathered} r = \sqrt{l_0}\,(\psi')^{-1/2}, \qquad r' = -\sqrt{l_0}\,(\psi')^{-3/2}\psi''/2, \\ r'' = -\sqrt{l_0}\,\Big((\psi')^{-3/2}\psi''' - 3(\psi')^{-5/2}\psi''^2/2\Big)/2. \end{gathered} \tag{5.6}$$

Substituting r'' from (5.6) and r from (5.4) into (5.5) and simplifying the result, we obtain

$$\{\psi\} = 2q(t) - 2\psi'^2. \tag{5.7}$$

Comparing formula (3.3) for $q(t)$ and the representation (2.7) for $B(t)$, we arrive at the SE (2.8) for the function ψ.

Corollary 1. *The auxiliary function τ defined by formula* (2.6) *has the following physical meaning: it coincides with the polar angle that describes the motion of the gas particle in the plane $z = 0$.*

This leads to the following important conclusion. The Schwarz equation (2.8) directly describes the kinematics of the gas particle.

To describe the motion of the gas in the HSV, one needs to know how to integrate the SE (2.8). This problem has been little studied [12, 13]; we give some new results (see Sec. 7).

6. Trajectory Equations. We integrate the equations of trajectories in the HSV for gas flows in which $\omega_\varphi = 0$. Then, only the first of integrals (1.9), (1.10) remains. The general solution (1.7) becomes

$$\cos\omega = \frac{F(\xi)}{\cos\tau\sin\theta} + \tan\,\tau\cot\,\theta, \tag{6.1}$$

where $\xi = r^2\tau'$ is the Lagrangian coordinate (1.8) in the HSV and F is an arbitrary function. The trajectory equations have the form

$$\frac{dr}{dt} = -\frac{1}{2}\,(\ln\tau')'r, \qquad \frac{d\theta}{dt} = \tau'\cos\omega, \qquad \frac{d\varphi}{dt} = \frac{\tau'\sin\omega}{\sin\theta}. \tag{6.2}$$

The first equation (6.2) specifies the area integral (3.4) or, in other terms, the representation for the Lagrangian coordinate ξ. Substituting expression (6.1) into the second equation (6.2), we obtain the following linear equation for $\psi = \cos\theta$:

$$\frac{d\psi}{d\tau} + \psi \tan\,\tau + \frac{F}{\cos\tau} = 0.$$

The general solution of this equation can be written as

$$\cos\theta = A_0 \cos\delta,$$

$$\delta = \tau + \alpha_0, \qquad A_0 = \sqrt{k_0^2 + F^2}, \qquad \alpha_0 = \arctan\,(F/k_0), \tag{6.3}$$

where k_0 is the integration constant. From (6.3) follows the inequality

$$0 \leqslant A_0 \leqslant 1.$$

Rewriting the third equation (6.2) as

$$d\varphi = \frac{\sin\omega}{\sin\theta}\, d\tau$$

and substituting into it the value

$$\sin\omega = \sqrt{1 - A_0^2}/\sin\theta,$$

obtained from (6.1) taking into account (6.3), we arrive at the integral

$$\varphi = \sqrt{1 - A_0^2}\int \frac{d\tau}{\sin^2\theta} = \sqrt{1 - A_0^2}\int \frac{d\delta}{1 - A_0^2\cos^2\delta},$$

which is taken in the form

$$\varphi = \arctan\,\Big(\tan\,\delta/\sqrt{1 - A_0^2}\Big). \tag{6.4}$$

Using the representations of Cartesian coordinates in terms of spherical coordinates, it is possible to simplify the expressions of $\cos\varphi$ and $\sin\varphi$ for the angle φ defined by (6.4). The final formulas are written as

$$x = r\sqrt{1 - A_0^2\cos^2\delta}\Big(\frac{1 - A_0^2}{1 - A_0^2 + \tan^2\delta}\Big)^{1/2}, \qquad y = r\sqrt{1 - A_0^2\cos^2\delta}\,\frac{\tan\,\delta}{(1 - A_0^2 + \mathrm{tg}^2\delta)^{1/2}},$$

$$z = rA_0\cos\delta, \qquad \delta = \tau + \alpha_0, \qquad 0 \leqslant A_0 \leqslant 1. \tag{6.5}$$

The function $\tau = \tau(t)$ is a solution of the SE (2.8).

7. Integration of the SE for Particular Values of γ. We consider an isothermal gas for which $\gamma = 1$. In this case, $p = S\rho$ and the velocity of sound $c^2 = S$ is conserved along the trajectory. Equation (2.8) becomes

$$\{\tau\} = \beta_0\tau' - 2\tau'^2, \tag{7.1}$$

where β_0 is specified by formula (2.5). We designate $\tau' = X$; then the numerator of the left side (7.1) is represented as

$$2XX'' - 3X'^2 = \frac{X^4}{X'}\Big(\frac{X'^2}{X^3}\Big)'.$$

Equation (7.1) is brought to the form

$$\Big(\frac{X'^2}{X^3}\Big)' = \frac{2X'}{X}\,(\beta_0 - 2X)$$

and is integrated once. The result is *the key equation* for $X = X(t)$:

$$X'^2 = X^3(\beta_0 \ln X^2 - 4X + C) \tag{7.2}$$

(C is an arbitrary constant).

Let $\gamma = 4/3$. Then, Eq. (2.4) becomes

$$\frac{2h'h''' - 3h''^2}{2h'^2} = \frac{\beta_0|h'|^{3/2}}{(1 + h^2)^{4/3}}. \tag{7.3}$$

The order of Eq. (7.3) is reduced by introducing a new function $X = X(h)$, so that $h' = X(h)$ (below, in Sec. 7 a prime denotes differentiation with respect to h). Substitution of X into (7.3) yields

$$2XX'' - X'^2 = 2\beta_0 |X|^{3/2}/(1+h^2)^{4/3}. \tag{7.4}$$

The left side of Eq. (7.4) is brought to the form

$$\frac{X^2}{X'}\Big(\frac{X'^2}{X}\Big)' = 2\beta_0\,\frac{|X|^{3/2}}{(1+h^2)^{4/3}}. \tag{7.5}$$

Introducing the new function

$$Z = X'/\sqrt{|X|} \tag{7.6}$$

we rewrite Eq. (7.5) as

$$(Z^2)' = 2\beta_0 Z/(1+h^2)^{4/3}. \tag{7.7}$$

After elimination of the derivative on the left side and simplifications, the above equation is brought to the form

$$\frac{dZ}{dh} = \frac{\beta_0}{(1+h^2)^{4/3}}$$

and is integrated twice. The first integration of (7.7) leads to the equation

$$Z = \beta_0(\Phi_1(h) + C_1), \tag{7.8}$$

where $(\Phi_1)'_h = (1+h^2)^{-4/3}$ and C_1 is the integration constant. According to (7.6), we have

$$Z = 2\,\frac{d}{dh}\,\sqrt{|X|}. \tag{7.9}$$

Hence, it is possible to integrate (7.8) once more. Taking into account (7.9), we obtain

$$\sqrt{|X|} = \beta_0(\Phi_2(h) + C_1 h + C_2)/2, \tag{7.10}$$

where $(\Phi_2)'_h = \Phi_1$ and C_2 is the integration constant. To eliminate the radical and the sign of the modulus, we raise (7.10) to the fourth power. The result is the *key equation*

$$\Big(\frac{dh}{dt}\Big)^2 = \alpha_0^4(\Phi_2(h) + C_1 h + C_2)^4, \qquad \alpha_0 = \frac{\beta_0}{2}. \tag{7.11}$$

The functions Φ_1 and Φ_2 are represented in terms of the hypergeometric Gauss function ${}_2F_1(a,b;c;x)$ [14] by the following formulas:

$$\Phi_1(h) = \int \frac{dh}{(1+h^2)^{4/3}} = \frac{3h}{2(1+h^2)^{1/3}} - \frac{1}{2}\,h\Phi(h),$$

$$\Phi_2(h) = (-3 + 12(1+h^2)^{2/3} - 4h^2\Phi(h))/8.$$

Here $\Phi(h) = {}_2F_1(1/2, 1/3; 2/3; -h^2)$.

Let $\gamma = 5/3$. A qualitative description of the gas flow in the HSV for this case is given in [6], where it is shown that the SE (2.4) splits into a first-order nonlinear ordinary differential equation and a second-order linear equation. The procedure of reducing the SE to the pair of equations described above can be simplified as follows.

Equation (2.4) is autonomous; the substitution $h' = X(h)$ reduces its order:

$$2XX'' - X'^2 = 2\beta_0 X^2/(1+h^2)^{5/3}. \tag{7.12}$$

Another substitution

$$|X| = Z^2 \tag{7.13}$$

reduces Eq. (7.12) to a second-order linear equation for the function $Z = Z(h)$:

$$\frac{d^2Z}{dh^2} - \frac{\beta_0 Z}{2(1+h^2)^{5/3}} = 0. \tag{7.14}$$

To eliminate the modulus sign in relation (7.13), we raise it to the square:

$$h'^2 = Z^4(h). \tag{7.15}$$

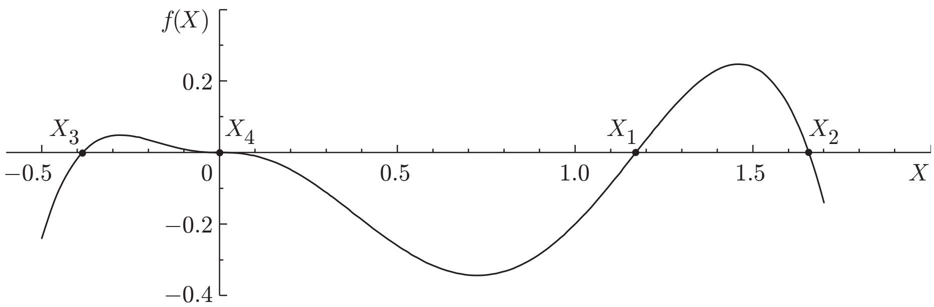

Fig. 2. Plot of the function $f(X)$ ($\beta_0 = 2.8$ and $C = 3.8$).

Thus, the SE (2.4) is equivalent to the following pair of equations: the second-order linear equation (7.14) and the first-order equation (7.15).

Naturally, the question arises of the relationship between the integrability of the SE (2.4) [or (2.8)] for the particular values of the exponent γ listed above and the existence of a nontrivial symmetry group of the this equation. This problem was studied by Cherevko, who proved that the SE (2.4) for $\gamma = 1$ and 5/3 admits a nontrivial group of tangential transformations. This group is responsible for the integration procedures described above. However, for $\gamma = 4/3$, no extension of the symmetry group of Eq. (2.4) was revealed and the nature of integration in this case remains unclear.

8. Analysis of Isothermal Motions of Gas ($\gamma = 1$). In the isothermal case, as can be seen from Sec. 7, the key equation (7.2), to which the SE (2.8) is reduced, has an especially simple form. However, the analysis of the motion is not trivial. There is an analogy between the solution obtained and the well-known periodic Sedov solution with a linear velocity field in the presence of a singularity [1]. The analysis of the solution is based on Ovsyannikov's theory of periodic gas motions [15].

In the HSV with $\gamma = 1$, a periodic kinematic configuration can exist in the plane specified by the initial data. However, the representation (2.7) implies the presence of a singularity in the solution — a density collapse. From a physical point of view, it makes sense to consider the solution on intervals that do not contain singular points.

The key equation (7.2) describing isothermal gas motions is written as

$$X'^2 = f(X), \tag{8.1}$$

where

$$f(X) = X^3\Phi(X), \qquad \Phi(X) = \beta_0 \ln X^2 - 4X + C. \tag{8.2}$$

The function $f(X)$ is plotted in Fig. 2. The plot has a cap generated by the roots X_1 and X_2 of the equation $\Phi(X) = 0$. Generally, it takes place if the following conditions are satisfied:

(a) X_1 and X_2 exit such that $f(X_1) = f(X_2) = 0$;

(b) $f'(X_1) > 0$, $f'(X_2) < 0$;

(c) the function $f(X)$ has a maximum at the point $X^* \in (X_1, X_2)$, i.e., $f'(X^*) = 0$ and $f''(X^*) < 0$.

These conditions, in turn, are ensured by the sufficient conditions of existence of the cap [8]. That is, if the function f depends also on a parameter $\lambda : f = f(X, \lambda)$ (in our case, this can be both C and β_0) and there exists a point $M(X_0, \lambda_0)$ at which $f(M) = 0$ and $f_X(M) = 0$ (condition 1) and $f'_\lambda(M) > 0$ and $f_{XX}(M) < 0$ (condition 2); then the cap exists for $\lambda > \lambda_0$, where λ is rather close to λ_0.

The presence of the cap on the plot guarantees the existence of a periodic solution of Eq. (8.1) [15].

Let us proof that conditions 1 and 2 are satisfied for a function f of the form (8.2). From condition 1 we have

$$X_0 = \beta_0/2, \qquad C_0 = 2\beta_0(1 + \ln 2 - \ln \beta_0). \tag{8.3}$$

Next, using (8.3), we obtain

$$f'_C(M) = X_0^3 = \beta_0^3/8, \qquad f_{XX}(M) = -2\beta_0/X_0^2 = -8/\beta_0. \tag{8.4}$$

Lemma 5. *The sufficient conditions for the existence of the cap of the function* (8.2) *are satisfied if*

$$\beta_0 > 0 \quad \textit{and} \quad C_0 = 2\beta_0(1 + \ln 2/\beta_0). \tag{8.5}$$

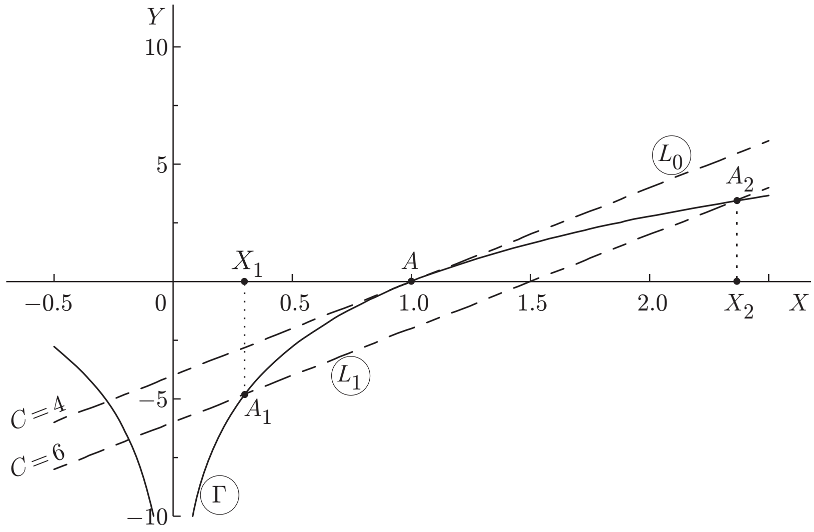

Fig. 3. Diagram of generation of the limiting period ($\beta_0 = 2$).

According to [15], if conditions 1 and 2 are satisfied, Eq. (8.1) has a periodic solution $X = X(t)$ with the period

$$\Pi = 2 \int_{X_1}^{X_2} \frac{dX}{\sqrt{f(X)}}. \tag{8.6}$$

The formulas describing the asymptotic representation of the solution and the period, have the form

$$X_\varepsilon(t) = X_0 - a \cos bt + O(\varepsilon^2); \tag{8.7}$$

$$\Pi_\varepsilon = 2\pi/b + O(\varepsilon). \tag{8.8}$$

Here $\varepsilon = \lambda - \lambda_0$ is a small parameter and the values of a and b are calculated from the value of the function f at the point M specified by conditions 1 and 2:

$$a^2 = 2f_\lambda(M)/|f_{XX}(M)|, \qquad 2b^2 = |f_{XX}(M)|. \tag{8.9}$$

Substituting the values from (8.4) into (8.9), we obtain

$$a = \beta_0^2/4\sqrt{2}, \qquad b = 2/\sqrt{\beta_0}. \tag{8.10}$$

Then, the limiting period calculated from (8.8) in accordance with (8.10) is equal to

$$\Pi = \pi\sqrt{\beta_0}.$$

The situation with the limiting period is illustrated by the following reasoning. Let us consider Fig. 3, which shows the curve Γ (a plot of the function $Y = 2\beta_0 \ln X$) and the straight lines $Y = 4X - C$ for two different values of the parameter C.

The straight line L_0 is tangent to the curve Γ at the point A and the straight line L_1 intersects it at the points A_1 and A_2. The projections of the points A_i onto the axis X specify the values of X_i that are the solutions of the equation $f(X) = 0$ defining the cap. Hence, in the limit, as the secant approaches the tangent $L_1 \to L_0$ and as $A_i \to A$, we obtain the limiting periodic motion with the period Π.

Numerical calculation of the dependence of the period Π on the parameter C shows that the function $\Pi = \Pi(C)$ increases monotonically, almost exponentially. Figure 4 gives a plot of this dependence for small C.

9. Equivalence Transformation of Eq. (8.1). Equation (8.1) admits the equivalence transformation

$$X = a\bar{X}, \qquad t = \bar{t}/a, \tag{9.1}$$

under which the constants β_0 and C are transformed as follows:

$$\bar{\beta}_0 = \beta_0/a, \qquad \bar{C} = (C + \beta_0 \ln a^2)/a. \tag{9.2}$$

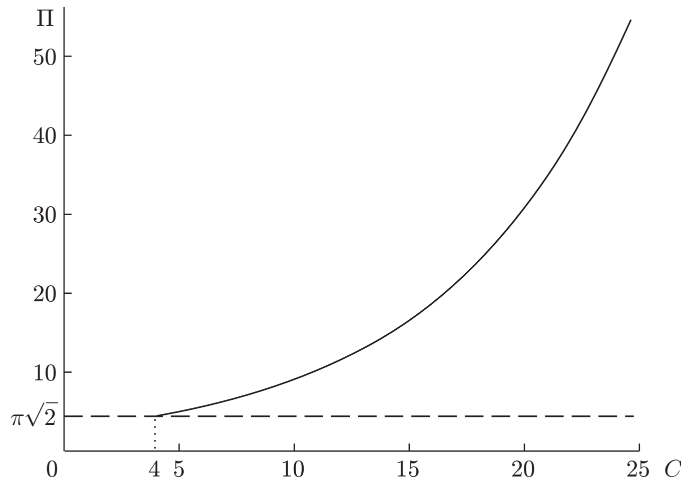

Fig. 4. Plot of the function $\Pi = \Pi(C)$ ($\beta_0 = 2$).

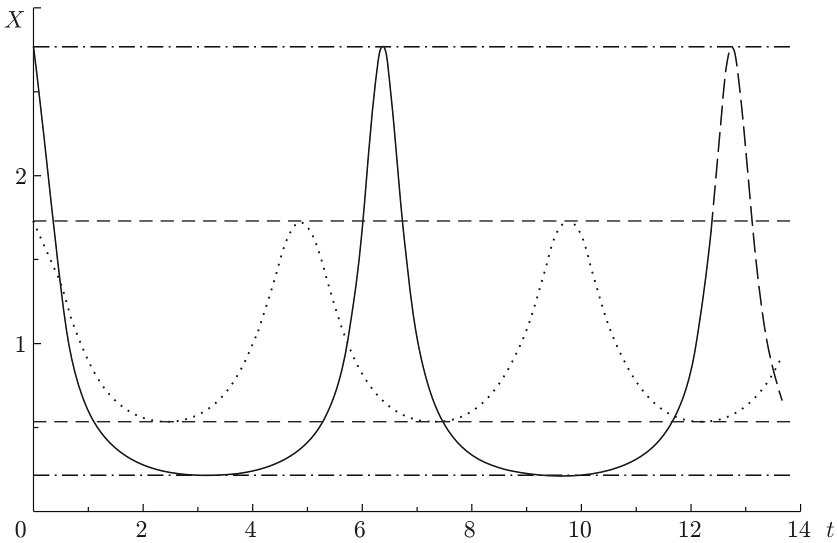

Fig. 5. Plot of $X = X(t)$ for various values of C ($\beta_0 = 2$).

Therefore, it suffices to study the dependence of the trajectory $X = X(t)$ only on one of the parameters (β_0 or C) because by the transformation (9.1), according to formulas (9.2), it is possible to fix one of them. It is convenient to set $\beta_0 = 2$ and to use the dependence of the solution only on C. This will be done below.

10. Closure of the Trajectory of the Solution $\boldsymbol{X = X(t)}$. We consider the solution $X = X(t)$ in the plane $\Pi : z = 0$, so that we can apply Lemma 4 and its corollary: Eq. (8.1) describes the kinematics of the gas particle. We introduce polar coordinates (r, ψ) in the plane Π; then the trajectory is defined by the following three relations:

$$X_t^2 = X^3(\beta_0 \ln X^2 - (4X - C)); \tag{10.1}$$

$$r^2 X = r_0^2 X_0; \tag{10.2}$$

$$\psi_t = X(t). \tag{10.3}$$

Lemma 6. *The integral curves corresponding to solutions* (10.1)–(10.3) *are located in the ring* $r \in [r_1, r_2]$.

Proof. Since $X \in [X_1, X_2]$, then, according to the integral (10.2), r varies in the closed interval $[r_1, r_2]$.

Figure 5 shows plots of the solution $X = X(t)$ for various values of the parameter C (the solid curve refers to $C = 7.0$ and the dotted curve to $C = 4.7$). As C increases, the value of X_1 decreases to zero and X_2 increases

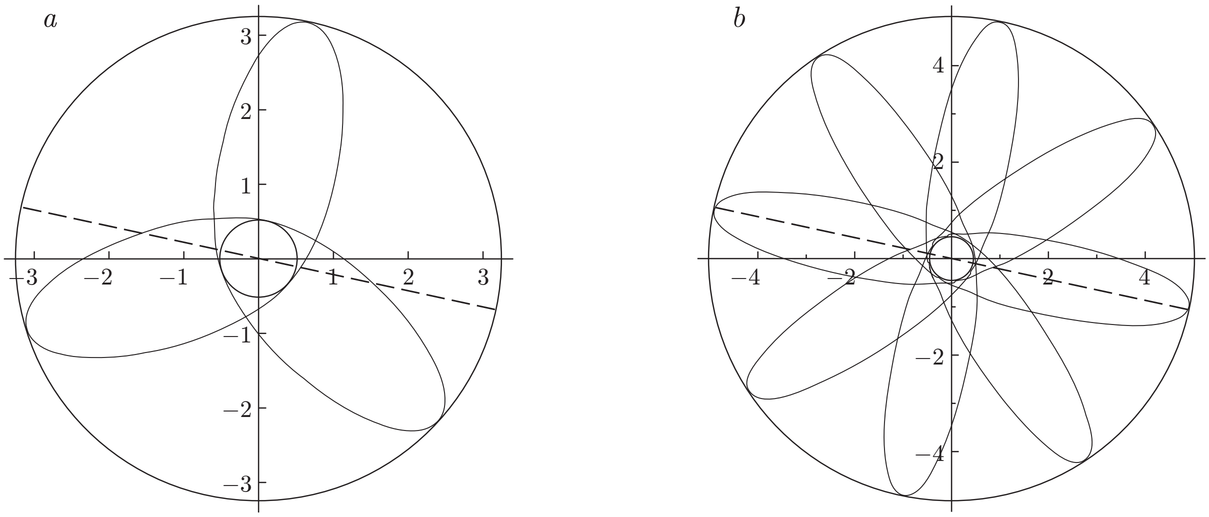

Fig. 6. Trajectories for $q = 2/3$ (a) and 5/8 (b).

without limit. In this case, the maximum of the function $f(X)$ increases; i.e., as t increases, the function $X(t)$ first increases and then decreases faster, which leads to sharpening and narrowing of the upward-directed peaks of the function $X(t)$.

The function $\psi = \psi(t)$, according to (10.3), has the form of a stepped curve. The quantity

$$\Psi = \int_{t_1}^{t_2} X(t)\, dt \tag{10.4}$$

defines the increase in the angle ψ in the time $\Pi = t_2 - t_1$ equal to the period $X(t)$. We note that the quantity Ψ is invariant under the equivalence transformation (9.1). For the existence of periodic solutions $X = X(t)$, it is necessary that the particle trajectory to be closed, possibly after several rotations around the center. Hence, the existence condition for the closed trajectories is given by

$$\Psi = 2\pi q, \qquad q = n/m \in \mathbb{Q}. \tag{10.5}$$

The number of rotations is specified by the numerator of the ratio n. The denominator m specifies the number of points at which the trajectory enters the external (internal) limiting circle (Lemma 6), i.e., it specifies the number of "lobes" on the trajectory. Numerical calculation shows that the quantity $\Psi = \Psi(C)$ is bounded from above and from below, at least, on a certain interval

$$\pi \leqslant \Psi \leqslant \pi\sqrt{2}. \tag{10.6}$$

The estimate (10.6) defines the possible choice of the parameter C and, hence, the number $q \in \mathbb{Q}$ in (10.5). Figure 6 shows the trajectories for two different values of q. On the dashed lines, according to (2.7), there is a density collapse for certain initial data.

For the trajectory configuration presented in Fig. 6a, Fig. 7 gives plots of the variations of the radius (solid curve), density (dashed curve), the radial gas particle velocity (dot-and-dashed curve), and the circular gas particle velocity (dotted curve) calculated according to formulas (2.2) and (2.3) and Eq. (8.1).

According to Lemma 2, each particle moves in its own plane. For two particles and $q = 2/3$, the possible spatial trajectory configuration is presented in Fig. 8.

11. Gas Jet Regime. The solution of the key equation (8.1) on the interval $X \in [X_3, X_4 = 0]$ (see Fig. 2) generates motion of the type of a gas jet. The equation $f(X) = 0$ has the root $X_4 = 0$ of multiplicity three; hence, according to the solution of Eq. (8.1)

$$t = \pm \int_{X_3}^{X_0} \frac{d\xi}{\sqrt{f(\xi)}}, \qquad X \to X_4, \tag{11.1}$$

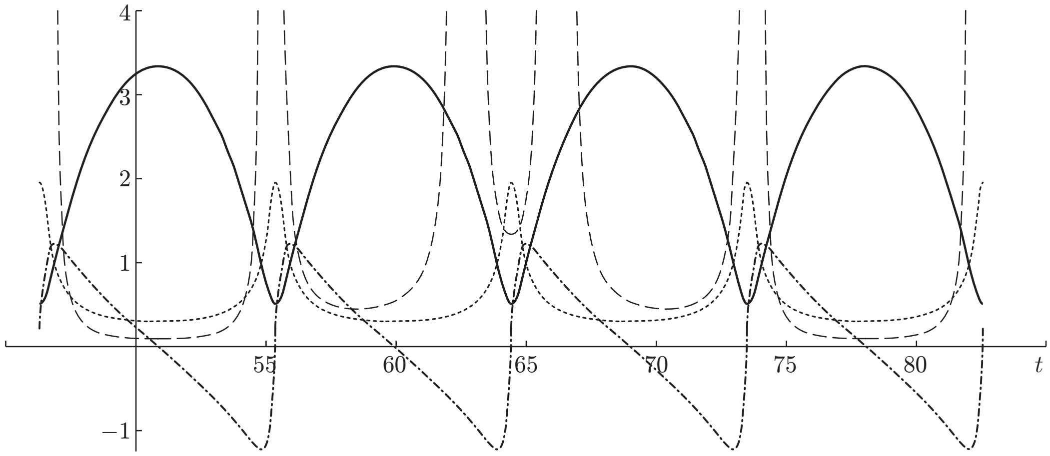

Fig. 7. Variation in the radius, density and circular and radial velocities along the trajectory in the case $q = 2/3$.

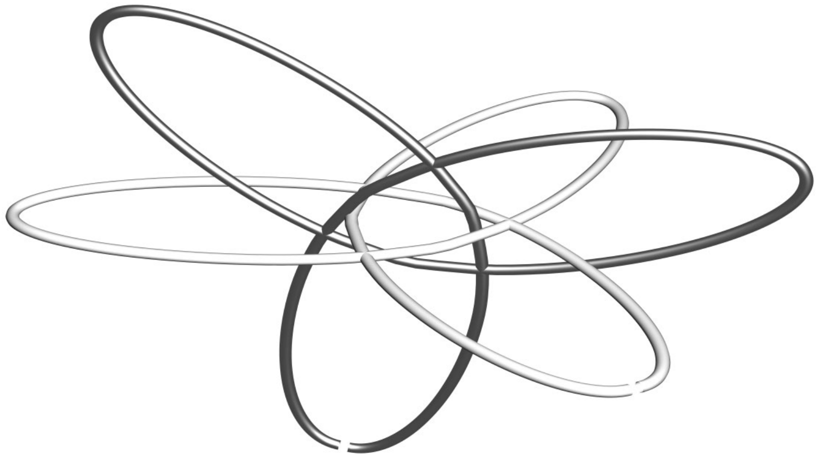

Fig. 8. Spatial trajectory configuration for two particles in the case $q = 2/3$.

the transition from the state X_3 to X_4 is performed for infinite time. Indeed, the improper integral on the right side of Eq. (11.1) is divergent and has a singularity of the form $\xi^{-3/2}$ at the point X_4 as $\xi \to 0$. Physically, this solution corresponds to the transition of the gas particle from the state with a finite $X_3 = \psi_3'$ to the state with $X_4 = \psi_4' = 0$, so that, according to the integral (10.2), we have: $\psi_4 \to \psi_{40} = \text{const}$ and $r_4 \to \infty$. The solution models the gas jet flowing to infinity or, vice versa, its arrival from infinity.

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